The two-parameter higher order differential calculus and curvature on a quantum plane.

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Abstract

We construct an associative differential algebra on a two-parameter quantum plane associated with a nilpotent endomorphism d in the two cases $d^2 = 0$ and $d^3 = 0$ ($d^2 \neq 0$). The correspondent curvature is derived and the related non commutative gauge field theory is introduced.

Keywords: Higher order non commutative differential calculi, two-parameter quantum plane, curvature, gauge theory.

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1 Introduction

Many authors [1, 2, 3, 4, 5, 6] have studied differential calculus with nilpotency $d^2 = 0$ on quantum spaces with one or two-parameter quantum group symmetry[7, 8].

Recently, there have been several attempts [9, 10, 11, 12, 13, 14] to generalize the classical exterior differential calculus to the one with exterior differential satisfying $d^N = 0$, $N \geq 3$. Such generalization is called q-differential algebra [13]. The latter is an associative unital N-graded algebra endowed with a linear endomorphism d satisfying $d^N = 0$ and the q-Leibniz rule:

$$d(uv) = (du)v + q^a u(dv),$$

where a is the grading of the element u and q is a primitive $N-$th root of unity.

Considering the particular generalization of classical exterior calculus corresponding to the case $d^3 = 0$, one replaces the condition $d^2 = 0$ by $d^3 = 0$ $(d^2 \neq 0)$, then one adds to the first order differential of coordinates $dx^1, dx^2, ...dx^n$ a set of second order differentials $d^2x^1, d^2x^2, ...d^2x^n$.

In the case of a quantum plane such generalization is possible and was studied in [15, 16, 17, 18]. The differential algebra satisfies the covariance property with respect to the quantum group symmetry of the quantum plane.

In the same spirit, we construct, in the present paper, a $GL_{p,q}(2)$ covariant associative differential algebra, in the two cases $d^2 = 0$ and $d^3 = 0$ $(d^2 \neq 0)$. We study the application of these differential calculi to the gauge field theory. In fact, we compute the correspondent curvatures, from which we extract the gauge field strength.

The paper is organized as follows:

In section 2 we give the basic definitions and results of the two-parameter quantum plane. Then we construct an associative differential calculus with $d^2 = 0$. We also introduce a generalization of the differential exterior derivative that satisfies, in stead of the $d^2 = 0$, the nilpotency condition $d^3 = 0$ $(d^2 \neq 0)$ in the sense of [5, 9].

In section 3, the nilpotent endomorphisms introduced in this work, are used to compute the correspondent curvature expressions on the quantum plane. Then the field strength which arises from this curvature is deduced.

2 Differential calculus on the two-parameter quantum plane.

2.1 Review of a two-parameter quantum plane.

The quantum plane [5, 7, 8] is an unital associative algebra generated by two non-commuting coordinates x and y satisfying the quadratic relation:

$$xy = qyx, \quad q \neq 0, 1, \quad q \in C. \tag{1}$$

This quantum plane admits a $GL_q(2)$ symmetry, in the sense that the relation (1) is invariant under its coaction. The main assumption behind this assertion is the commutation of the

coordinates of the quantum plane and the generators of the quantum group $GL_q(2)$ (i.e. the entries of the matrix $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, as a generic element of $GL_q(2)$).

However, if one relaxes this assumption by assuming generic non-commutation relations between the space coordinates (x, y) and group generators (a, b, c, d), the $GL_q(2)$ symmetry is extended to a $GL_{p,q}(2)$ one.

Let us recall [4, 19] that $GL_{p,q}(2)$ is a quantum group generated by a, b, c, d obeying the following relations:

$$\begin{aligned} ab &= pba \qquad & cd = pdc \qquad & pbc = qcb \\ ac &= qca \qquad & bd = qdc \qquad & ad - da = (p - \frac{1}{q})bc\,, \end{aligned} \tag{2}$$

for some non zero complex p, q with $qp \neq -1$. For latter use and basing on the results of [15, 16] the assumption of non-commutativity between space coordinates and $GL_{p,q}(2)$ generators takes the form:

$$\begin{aligned} xa &= ax \qquad & xb = \frac{q}{p}bx \qquad & xc = cx \qquad & xd = \frac{q}{p}dx \\ ya &= \frac{q}{p}ay \qquad & yb = \left(\frac{q}{p}\right)^2 by \qquad & yc = \frac{q}{p}cy \qquad & yd = \left(\frac{q}{p}\right)^2 dy, \end{aligned} \tag{3}$$

for further details concerning this construction see [4, 16, 19].

In the limit $p \to q$, we recover the commutativity between coordinates and generators, and $GL_{p,q}(2)$ reduces to $GL_q(2)$.

2.2 Quantum differential algebra, $d^2 = 0$.

The aim of this section is to construct an associative differential algebra ${}^2\Omega_{p,q} = \{x, y, dx, dy\}$, on the two-parameter quantum plane. The covariance of this differential algebra, with respect to $GL_{p,q}(2)$, is ensured if we proceed as in [5, 6, 15].

We start by defining the differential operator d satisfying

$$d(x) = dx, \qquad d(y) = dy, \qquad d(1) = 0.$$

More generally the operator d acts as follows:

$$d : \Omega^n \to \Omega^{n+1},$$

where Ω^n is the space of forms with degree n; Ω^0 being the algebra of functions defined on the quantum plane.

The operator d must also obey the usual properties of:

i/ Linearity:

$$d(\alpha z + \beta z') = \alpha d(z) + \beta d(z'),$$

where z, $z' \in {}^2\Omega_{p,q}$ and α, β are either complex numbers or the generators $\{a, b, c, d\}$ of $GL_{p,q}(2)$.

ii/ Nilpotency:

$$d^2 = 0$$

iii/ Liebniz rule:

$$d(uv) = (du)v + (-1)^n u(dv), \tag{4}$$

where $u \in \Omega^n$.

Moreover, the commutation relations between the elements of ${}^2\Omega_{p,q}$ must be covariant under $GL_{p,q}(2)$. Namely, one can write a priori xdx, dy, , ydx and ydy in terms of $(dx)x$, $(dy)x$, $(dx)y$ and $(dy)y$. Imposing the covariance of the obtained relations under $GL_{p,q}(2)$, associativity of expressions such as $(xdx)dy = x(dxdy)$ and differentiating (1) yields two possible (mutually related) differential calculi. One of these calculi was given in [16]. The other, which we will use in the sequel to construct gauge field theory, is given by the following commutation relations

$$\begin{aligned} x\,dx &= qp\,dx\,x & \qquad x\,dy &= q\,dy\,x + (qp-1)dx\,y \\ y\,dy &= qp\,dy\,y & \qquad y\,dx &= p\,dx\,y \\ dy\,dx &= -p\,dx\,dy & \qquad (dx)^2 &= (dy)^2 = 0, \end{aligned} \tag{5}$$

The associative differential calculus on the one-parameter quantum plane [2, 3, 5, 15] can be obtained as a limiting case from the two-parameter one by taking $p \rightarrow q$.

Furthermore, the differential exterior operator d is realized as in the standard way:

$$d = dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} = dx\partial_x + dy\partial_y, \tag{6}$$

where ∂_x and ∂_y are respectively, the partial derivatives in the directions x and y.

Consistency conditions [2] then yields:

$$\begin{aligned} \partial_x x &= 1 + qpx\partial_x + (qp-1)y\partial_y & \qquad \partial_y x &= qx\partial_y \\ \partial_x y &= py\partial_x & \qquad \partial_y y &= 1 + qpy\partial_y \\ \partial_x\partial_y &= p\partial_y\partial_x\,. & & \end{aligned} \tag{7}$$

Latter, we will apply this associative covariant differential calculus to construct a non commutative gauge field theory on the two-parameter quantum plane.

The above algebra ${}^2\Omega_{p,q}$ can be generalized to the one associated with a nilpotent endomorphism d satisfying $d^3 = 0$. This generalization is defined and studied in the following section.

2.3 Differential algebra on the two-parameter quantum plane with $d^3 = 0$.

3-nilpotent differential calculi on specific non commutative spaces (one and two-parameter quantum plane and superplane) have been constructed in different papers [15, 16, 17, 18].

In a more general context many authors [11, 13, 20] have studied generalized differential calculi with $d^N = 0$ and gave some examples related to theoretical physics. Here we shall not give an exposition of these generalizations, but, instead, we shall follow a point of view which

was illustrated in [15, 16]. Indeed, one can always define the differential operator d satisfying linearity, nilpotency $d^3 = 0$ ($d^2 \neq 0$) and the j-Leibniz rule:

$$d(uv) = (du)v + (j)^n u(dv), \qquad j^3 = 1, \tag{8}$$

where $u \in \Omega^n$, the space of n-forms defined on the two-parameter quantum plane.

Following the same method adopted in [15, 16], we obtain the two-parameter associative covariant differential algebra ${}^3\Omega_{p,q} = \{x, y, dx, dy, d^2x, d^2y\}$. However in [16] (not in [15]) the obtained differential algebra was not associative and we sketched an adequate way leading to an associative one.

In the present paper, the associativity of the differential algebra is required in order to construct a gauge field theory. This point will become clear in the next sections.

Basing on the previous discussion it is easy to show that there will be an extra constraint on the two parameters p and q

$$qp = j^2, \tag{9}$$

which arises from the rquirement of the associativity property. And the commutation relations between the differential generators of ${}^3\Omega_{p,q}$ are given by:

First order

$$\begin{aligned} x\,dx &= qp\;dx\,x & \qquad x\,dy &= q\;dy\,x + (qp-1)dx\,y \\ y\,dy &= qp\;dy\,y & \qquad y\,dx &= p\;dx\,y \end{aligned} \tag{10}$$

Second order

$$\begin{aligned} x\,d^2x &= qp\;d^2x\,x & \qquad x\,d^2y &= q\;d^2y\,x + (qp-1)d^2x\,y \\ y\,d^2y &= qp\;d^2y\,y & \qquad y\,d^2x &= p\;d^2x\,y \\ dx\,dy &= q\;dy\,dx & \qquad dy\,dx &= jp\;dx\,dy \end{aligned} \tag{11}$$

Third order

$$\begin{aligned} dx\,d^2x &= j\;d^2x\,dx & \qquad dx\,d^2y &= j^2q\;d^2y\,dx + j^2(qp-1)\;d^2x\,dy \\ dy\,d^2y &= j\;d^2y\,dy & \qquad dy\,d^2x &= j^2p\;d^2x\,dy \end{aligned} \tag{12}$$

Fourth order

$$d^2x\,d^2y = q\;d^2yd^2x. \tag{13}$$

Using the realization of the differential operator d :

$$d = dx\partial_x + dy\partial_y,$$

one can prove that:

$$\begin{aligned} \partial_x x &= 1 + qpx\partial_x + (qp-1)y\partial_y & \qquad \partial_y x &= qx\partial_y \\ \partial_x y &= py\partial_x & \qquad \partial_y y &= 1 + qpy\partial_y \\ \partial_x\partial_y &= p\partial_y\partial_x & \qquad (dx)^3 &= (dy)^3 = 0. \end{aligned} \tag{14}$$

The natural requirement to recover the one-parameter differential calculus when $p \to q$ is preserved. In fact, in this limit one has $q = j$, due to (9), and all the results reduce to their counterparts obtained in [15]; especially, the two-parameter quantum plane becomes a reduced quantum plane.

As a physical application of these differential calculi '$d^2 = 0$' and '$d^3 = 0$' we construct, in the sections below, the corresponding gauge field theory.

3 Covariant derivative and curvature on a two-parameter quantum plane.

3.1 '$d^2 = 0$' case.

Here, our main purpose is to define the covariant derivative; as in the ordinary case, this enables us to introduce the notion of curvature, the components of which, will be discussed by comparing them to the ordinary ones.

So, as in the commutative case, the covariant differential acting on a field $\Phi(x, y)$ is defined by:

$$D\Phi(x, y) = d\Phi(x, y) + A(x, y)\Phi(x, y), \tag{15}$$

where $\Phi(x, y)$ is a function on the two-parameter quantum plane and the gauge field $A(x, y)$ is a 1-form valued in the associative algebra of functions on the two-parameter quantum plane.

We notice that the bimodule structure of the algebra of functions on the two-parameter quantum plane over ${}^2\Omega_{p,q}$ is assumed.

As usual, the covariant differential D must satisfy

$$DU^{-1}\Phi(x, y) = U^{-1}D\Phi(x, y), \tag{16}$$

where U is an endomorphism defined on the algebra of functions over the two-parameter quantum plane.

From (16) it follows that the gauge field transforms as:

$$A(x, y) \to U^{-1}A(x, y)U + U^{-1}dU. \tag{17}$$

The curvature is defined through:

$$D^2\Phi(x, y) = (dA(x, y) + A(x, y)A(x, y))\,\Phi(x, y) := R\Phi(x, y). \tag{18}$$

The gauge field $A(x, y)$ being a 1-form, generally takes the form:

$$A(x, y) = dx\ A_x(x, y) + dy\ A_y(x, y), \tag{19}$$

where the component $A_x(x, y)$ and $A_y(x, y)$ are functions on the two-parameter quantum plane.

$Eq(19)$, together with the differential realization of d (6) and taking account of the associativity of ${}^2\Omega_{p,q}$, allows to rewrite the curvature R:

$$
\begin{aligned}
R &= dxdy\left(\frac{1}{p}\partial_x A_y(x,y) - \partial_y A_x(x,y)\right) + dxA_x(x,y)dxA_x(x,y) + \\
&\quad + dxA_x(x,y)dyA_y(x,y) + dyA_y(x,y)dxA_x(x,y) + dyA_y(x,y)dyA_y(x,y). \qquad (20)
\end{aligned}
$$

One has to express the curvature written above in terms of 2-forms constructed from the basic generators of the differential algebra ${}^2\Omega_{p,q}$. Since we are dealing with a non-commutative space (two-parameter quantum plane), this task is not straightforward. In fact, the non-commutativity prevents us from rearranging the different terms in (20) adequately. To overcome this technical difficulty we require that the components of the gauge field, $A_x(x,y)$ and $A_y(x,y)$, are expressed as formal power series of the space coordinates:

$$
\begin{aligned}
A_x(x,y) &= a_{n,m}y^n x^m \qquad (21)\\
A_y(x,y) &= b_{n,m}y^n x^m,
\end{aligned}
$$

where n and m are integers. Using the formulas $(1),(5),(7),(20),(21)$ and after technical computations, an explicit expression of curvature arises:

$$
R = dxdy\ \left[F_{xy}^{\frac{1}{p}} - \frac{1}{p}\ \ \alpha^{n,m}b_{n,m}y^{n+1}x^{m-1}A_y(x,y)\right], \qquad (22)
$$

where

$$
F_{xy}^{\frac{1}{p}} = \frac{1}{p}\partial_x A_y(x,y) - \partial_y A_x(x,y) + A_x(qx,qpy)A_y(x,y) - \frac{1}{p}A_y(qpx,py)A_x(x,y) \qquad (23)
$$

and

$$
\alpha^{n,m} = p^n q^{m-1}\ \ ((qp)^m - 1)\,. \qquad (24)
$$

It is easy to check that the deformed field strength $F_{xy}^{\frac{1}{p}}$ satisfy a p-antisymmetry property:

$$
F_{yx}^{p} = -p\ F_{xy}^{\frac{1}{p}}\ .
$$

If one takes the limit $p \to q = j$, the curvature expression obtained in [15] is recovered. More interesting is the limit $p \to q \to 1$, where one expects to recover the commutative case. This is valid:

$$
F_{xy}^{\frac{1}{p}} \to F_{xy}\ \ \text{since}\quad \alpha^{n,m} \to 0.
$$

This remark allows the interpretation of the supplementary terms $\left(-\frac{1}{p}\ \alpha^{n,m}b_{n,m}y^{n+1}x^{m-1}A_y(x,y)\right)$, appearing in (22), as a direct consequence of the non-commutativity of the space.

3.2 '$d^3 = 0$' case.

In this section we extend the results of the previous one in the sense that we will use the differential calculus '$d^3 = 0$'.

The same definition (15) of the covariant differential as well as properties (16), (17), (19) are preserved in this case [11].

However, the curvature is no more a 2-form defined in (18) but is a 3-form [10, 11]:

$$D^3\Phi(x,y) := R\Phi(x,y). \tag{25}$$

Direct computations show that R is given by:

$$\begin{aligned} R &= d^2A(x,y) + dA^2(x,y) + A(x,y)dA(x,y) + A^3(x,y) \\ &= d^2A(x,y) + dA(x,y)A(x,y) - j^2A(x,y)dA(x,y) + A^3(x,y). \end{aligned} \tag{26}$$

As in the previous case ($d^2 = 0$), we will use the formal power series expansion (21) of the gauge field. Then general formulas (given in appendix A) can be obtained using (1), (10) − (14). These formulas simplify the computation of the curvature R and permit to write it in terms of the basic generators of ${}^3\Omega_{p,q}$:

$$\begin{aligned} R = \; & jd^2xdyF^p_{xy} + d^2ydxF^{jq}_{yx} + (dx)^2dy(R_{xxy} + R_{xyx} + R_{yxx}) + (dy)^2dx(R_{yyx} + R_{yxy} + R_{xyy}) \\ & -jpd^2xdyA^b_0(x,y)A_y(x,y) + d^2ydxA^b_0(x,y)A_y(x,y) \\ & +(dx)^2dy\Big[\Big\{j^2pA_2(x,y) + p^2A^b_1(x,y) + (jp)^2A^b_0(qpx,py)A^b_0(x,y)\Big\}A_y(x,y) \\ & +jpA_x(q^2px,qp^2y)A^b_0(x,y)A_y(x,y) + q(jp)^2A^b_0(qpx,py)A_x(qx,qpy)A_y(x,y) \\ & +(jp)^2A^b_0(qpx,py)A_y(qpx,py)A_x(x,y) \\ & +\Big\{(1+jp^2)A_0(qx,qpy) + (jp)^2A^b_3(x,y)\Big\}A_y(qx,qpy)A_y(x,y) \\ & +(jp)^2A_y((qp)^2x,p^2y)A_0(x,y)A_y(x,y) \\ & -\Big\{(1+jp^2)A_0(qx,qpy) + (jp)^2A^b_3(x,y)\Big\}\partial_yA_y(x,y) \\ & -A^b_0(qpx,py)\Big\{q(jp)^2\partial_yA_x(x,y) + qp^3\partial_xA_y(x,y)\Big\}\Big] \\ & +(dy)^2dx\Big[jA^b_2(x,y)A_y(x,y) + (q+p)A^b_0(qx,qpy)A_y(qx,qpy)A_y(x,y) \\ & +A_y(q^2px,qp^2y)A^b_0(x,y)A_y(x,y) - (q+p)A^b_0(qx,qpy)\partial_yA_y(x,y)\Big], \end{aligned} \tag{27}$$

where the field strength is given by

$$\begin{aligned} F^p_{xy} &= p\partial_xA_y(x,y) - \partial_yA_x(x,y) + A_x(qx,qpy)A_y(x,y) - pA_y(qpx,py)A_x(x,y) \\ F^{jq}_{yx} &= jq\partial_yA_x(x,y) - \partial_xA_y(x,y) + A_y(qpx,py)A_x(x,y) - jqA_x(qx,qpy)A_y(x,y). \end{aligned} \tag{28}$$

The components $R_{ijk}, i,j,k = x,y$, and the functions $A_0, A^b_0, A^b_1, A^b_2, A^b_3$ are given in appendix B.

The antisymmetric property of the field strength F_{xy} in the commutative case, is replaced by a p-antisymmetry:

$$F^{p}_{xy} = -pF^{jq}_{yx} \tag{29}$$

In the limit $p \to q$ the components $F^{p}_{xy}, F^{jq}_{yx}, R_{ijk}$ and $A_0, A^b_0, A^b_1, A^b_2, A^b_3$ reduce to their one-parameter counterparts obtained in [15]; then the curvature will be identical to the one in [15].

The expression of the curvature components (31) in appendix B, and the deformed field strength (28) are formally the same as those obtained by [10, 11]. The supplementary terms, where the functions $A_0, A^b_0, A^b_1, A^b_2, A^b_3$ appear, can be interpreted as a direct consequence of the non-commutativity property of the space.

Moreover, compared with the case $d^2 = 0$, the curvature expression contains additional terms R_{ijk}. These terms can be interpreted as a generic consequence of the generalization of the differential calculus $d^2 = 0$ to a higher order $d^3 = 0$.

4 Conclusion

In this work we have constructed associative differential calculi $d^2 = 0,\ d^3 = 0$ on the two-parameter quantum plane. The notion of covariance of these differential calculi is also ensured and we have shown that there is a quantum group, $GL_{p,q}(2)$, behind this covariance.

As an application, we have constructed a gauge field theory based on these differential calculi. The limit $p \to q$ was also studied in the two cases $d^2 = 0,\ d^3 = 0$ and yields the results of [15].

Appendix A

In this appendix we give some general formulas which are useful in the computation of the curvature components.

$$\begin{aligned}
&x^n\, dx = (qp)^n\, dx\, x^n && x^n\, dy = q^n\, dy\, x^n + q^{n-1}\Big((qp)^n - 1\Big)\, dx\, y\, x^{n-1} \\
&y^n\, dx = p^n\, dx\, y^n && y^n\, dy = (qp)^n\, dy\, y^n \\
&\partial_x(y^n x^m) = p^n \left(\frac{1-(qp)^m}{1-qp}\right) y^n\, x^{m-1} && \partial_y(y^n x^m) = \left(\frac{1-(qp)^n}{1-qp}\right) y^{n-1}\, x^m \\
&A_z(x,y)\, dx = dx\, A_z(qpx, py) && A_z(x,y)\, dy = dy\, A_z(qx, qpy) + dx\, \alpha^{n,m} c_{n,m}\, y^{n+1}\, x^{m-1} \\
&\partial_x A_z(x,y)\, dx = dx\, \partial_x A_z\mid_{(qpx,py)} && \partial_x A_z(x,y)\, dy = dy\, \partial_x A_z\mid_{(qx,qpy)} + dx\, \beta^{n,m}_x c_{n,m}\, y^{n+1}\, x^{m-2} \\
&\partial_y A_z(x,y)\, dx = dx\, \partial_y A_z\mid_{(qpx,py)} && \partial_y A_z(x,y)\, dy = dy\, \partial_y A_z\mid_{(qx,qpy)} + dx\, \beta^{n,m}_y c_{n,m}\, y^{n}\, x^{m-1},
\end{aligned} \tag{30}$$

where $z = x, y$ and $c_{n,m} = a_{n,m}$ for $z = x$ or $c_{n,m} = b_{n,m}$ for $z = y$.

Appendix B

In this appendix we give the explicit expression of the curvature components appearing in (24)

$$
\begin{aligned}
R_{xxy} &= (jp)^2\ \partial_x\partial_x A_y(x,y) + j\ \partial_x A_x\Big|_{(qx,qpy)}\ A_y(x,y) - jp\ A_x(q^2px,qp^2y)\,\partial_x\,A_y(x,y)+\\
&\qquad A_x(q^2px,qp^2y)\,A_x(qx,qpy)\,A_y(x,y)\\
R_{yxx} &= \partial_y\partial_x A_x(x,y) + j^2p\ \partial_y A_x\Big|_{(qpx,py)}\ A_x(x,y) - (jp)^2\ A_y((qp)^2x,p^2y)\,\partial_x\,A_x(x,y)+\\
&\qquad (jp)^2\ A_y((qp)^2x,p^2y)\,A_x(qpx,py)\,A_x(x,y)\\
R_{xyx} &= jp\ \partial_x\partial_y A_x(x,y) + p^2\ \partial_x A_y\Big|_{(qpx,py)}\ A_x(x,y) - A_x(q^2px,qp^2y)\,\partial_y\,A_x(x,y)+\\
&\qquad jp\ A_x(q^2px,qp^2y)\,A_y(qpx,py)\,A_x(x,y) \qquad (31)\\
R_{yyx} &= q^2\ \partial_y\partial_y A_x(x,y) + j\ \partial_y A_y\Big|_{(qpx,py)}\ A_x(x,y) - q\ A_y(q^2px,qp^2y)\,\partial_y A_x(x,y)+\\
&\qquad A_y(q^2px,qp^2y)\,A_y(qpx,py)\,A_x(x,y)\\
R_{yxy} &= q\ \partial_y\partial_x A_y(x,y) + jq^2\ \partial_y A_x\Big|_{(qx,qpy)}\ A_y(x,y) - jqp\ A_y(q^2px,qp^2y)\,\partial_x A_y(x,y)+\\
&\qquad q\ A_y(q^2px,qp^2y)\,A_x(qx,qpy)\,A_y(x,y)\\
R_{xyy} &= \partial_x\partial_y A_y(x,y) + j^2q^2p\ \partial_x A_y\Big|_{(qx,qpy)}\ A_y(x,y) - q^2\ A_x(q^2x,(qp)^2y)\,\partial_y A_y(x,y)+\\
&\qquad q^2\ A_x(q^2x,(qp)^2y)\,A_y(qx,qpy)\,A_y(x,y)\,.
\end{aligned}
$$

The functions $A_0, A_0^b, A_1^b, A_2^b, A_3^b$ are given by

$$
\begin{array}{ll}
A_0 = \alpha^{n,m} a_{n,m}\ y^{n+1}\,x^{m-1} & A_0^b = \alpha^{n,m} b_{n,m}\ y^{n+1}\,x^{m-1}\\
A_1^b = \beta_x^{n,m} b_{n,m}\ y^{n+1}\,x^{m-2} & A_2^b = \beta_y^{n,m} b_{n,m}\ y^{n}\,x^{m-1}\\
A_3^b = \alpha^{n,m} b_{n,m} q^{m-2}p^{n+1}\Big((qp)^{m-1}-1\Big)\ y^{n+2}\,x^{m-2}, &
\end{array} \qquad (32)
$$

where the coefficients $\alpha^{n,m}$, $\beta_x^{n,m}$ and $\beta_y^{n,m}$ are given by:

$$
\begin{aligned}
\alpha^{n,m} &= q^{m-1}p^n\ \Big((qp)^m-1\Big)\\
\beta_x^{n,m} &= q^{m-2}p^{2n}\Big((qp)^{m-1}-1\Big)\left(\frac{1-(qp)m}{1-qp}\right)\\
\beta_y^{n,m} &= q^{m-1}p^{n-1}\Big((qp)^m-1\Big)\left(\frac{1-(qp)^n}{1-qp}\right)
\end{aligned} \qquad (33)
$$

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